Partition of Networks

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1 Purpose

To give a relation between structure of a network and eigenvalues of the Laplacian of the network. The eigenvalues relate to the spectrum of a Hamiltonian in a Tight-binding Model.

$G = (V, E)$: a network (V: the node set, E: the bond set)

2 Laplacian of Network

The Laplacian of the network gives the kinetic energy of a particle hopping between the nodes. The Laplacian is a $n \times n$ matrix (n is the number of nodes), which is a discrete analog of the second derivative of functions on V.

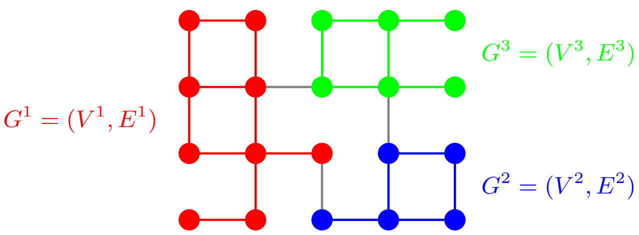
Let $\lambda_k(G)$ be the k-th smallest eigenvalue ($k = 1, 2, \dots n$) of the Laplacian, which represents k-th energy level. The eigenfunction with respect to $\lambda_k(G)$ represents the eigenstate of the k-th energy level.

3 Connectivity of Network

The k-way expansion constant $h_k(G)$ is a strength of connectivity of G with respect to partitions into k subnetworks.

$$h_k(G) := \min \left\{ \max_{i=1,2,\dots,k} \frac{|\partial V^i|}{|V^i|} : V = \bigsqcup_{i=1}^{k} V^i, V^i \neq \emptyset \right\},$$

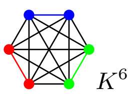
where ∂F is the set of the bonds connecting F and $V - F$.



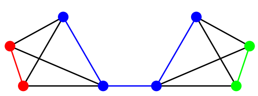
4 Examples

Example 1. The number of the connected component of G is k if and only if $\lambda_k(G) = 0$ and $\lambda_{k+1}(G) > 0$ if and only if $h_k(G) = 0$ and $h_{k+1}(G) > 0$.

Example 2 (Complete graph)**.** $\lambda_k(K^{kn}) = kn$ and $h_k(K^{kn}) = (k-1)n$ for $n, k \geq 2$. Connectivity of complete graphs are very strong.

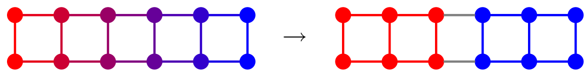


Example 3. Let $G_{n,m} := (V_{K^n} \cup V_{K^m}, E_{K^n} \cup E_{K^m} \cup \{vw\})$ for $n, m \in \mathbb{N}$, where $v \in V_{K^n}$ and $w \in V_{K^m}$. Then $h_2(G_{n,m}) = 1/\min\{n, m\}$. For $n \in \mathbb{N}$, $h_3(G_{2n,2n}) = n \gg h_2(G_{2n,2n}) = 1/2n$.



5 Relation between λ_k and h_k

The eigenfunction with respect to $\lambda_1(G)$ is constant. The other eigenfunctions have positive and negative values (and zero). Hence such eigenfunctions gives a partition into positive value node set and non-positive value node set.



Using such partitions, we can give a k-partition from eigenfunctions of $\lambda_2(G), \dots, \lambda_k(G)$ which is similar to the k-partition with respect to $h_k(G)$.

Theorem 4 (Lee-Gharan-Trevisan [1])**.** *There is a constant $C > 0$ such that*

$$\frac{\lambda_k(G)}{2\deg(G)} \leq h_k(G) \leq Ck^2 \deg(G)\sqrt{\lambda_k(G)}$$

for every connected networks G and every $k = 2, \dots, n$, where $\deg(G)$ is the maximum number of nodes around one node.

6 Spectral Gap and Partition of Network

In [2], for G with $h_k(G) \ll h_{k+1}(G)$ we gave a k-partition reflecting a geometry of G. In [3], they give better k-partition under an weaker condition.

Theorem 5 (Gharan-Trevisan [3])**.** *If $h_{k+1}(G) > (1+\epsilon)h_k(G)$ for some $0 < \epsilon < 1$, then there exists a k-partition $\{G^i = (V^i, E^i)\}_{i=1}^k$ of G satisfying*

$$\frac{\epsilon h_{k+1}(G)}{14k} \leq h_2(G^i), \qquad \frac{|\partial V^i|}{|V^i|} \leq k h_k(G)$$

for all $i = 1, 2, \dots, k$.

The left inequality means the strength of connectivity of each subnetworks is estimated by using the spectral gap between $h_k(G)$ and $h_{k+1}(G)$, because the spectral gap is less than $\epsilon h_{k+1}(G)$ when ϵ is appropriate. The right inequality means the degree in separation of G_i form other subnetworks is estimated by using $h_k(G)$.

Using Theorem 4, we can translate this theorem into inequalities between eigenvalues.

7 Future Work

The above theorems are meaningful only for small k. But in some situations we need a relation between a geometry of network and spectral gap near $k = n/2$. We research this now.

References

[1] J. R. Lee, S. O. Gharan, and L. Trevisan, STOC 2012.

[2] M. Tanaka, arXiv:1112.3434.

[3] S. O. Gharan and L. Trevisan, SODA 2014.